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PARTIAL GEOMETRIC DESIGNS AND TWO-CLASS PARTIALLY BALANCED DESIGNS

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It is shown that a partial geometric design with parameters (r, k, t, c) satisfying certain conditions is equivalent to a two-class partially balanced incomplete block design. This generalizes a result concerning partial geometric designs and balanced incomplete block designs.

1. Introduction

Let $D(v, b, r, k)$ be any tactical configuration. Here v denotes the number of points of D, b the number of blocks (= certain subsets of the points of D), r the number of blocks containing any point of D and k the block size of D.

Let N denote the usual $v \times b$ (0, 1) incidence matrix of D. The configuration D is said to be a partial geometric design with parameters (r, k, t, c), $t \geq 1$ if in addition to

$$NJ = rJ, \qquad JN = kJ, \tag{1}$$

we have

$$NN^tN = (r + k + c - 1 - t)N + tJ, \tag{2}$$

where J denotes the all one matrix of the appropriate size and N^t is the transpose of N. Partial geometric designs were introduced in [2] and were used to generalize the Hall–Connor embedding theorem. For a spectral characterization of partial geometric designs we refer to [1].

Let N denote the incidence matrix of a partial geometric design (r, k, t, c). Put

$$P = NN^t - rI. \tag{3}$$

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Then P is a non-negative, symmetric $v \times v$ integral matrix with zero trace. Moreover (2) can be rewritten as

$$PN = (k + c - 1 - t)N + tJ. \tag{4}$$

Let E be any two-class partially balanced incomplete block design (PBIBD) with parameters $(v, b, r, k, n_1, \lambda_1, \lambda_2)$ based on an association scheme for which Q is the matrix of first associates. Let N be the usual $v \times b$ (0, 1) incidence matrix of E. Then we have

$$NJ = rJ, \qquad JN = kJ, \tag{5}$$

and

$$NN^t(r - \lambda_2)I + (\lambda_1 - \lambda_2)Q + \lambda_2 J. \tag{6}$$

We recall that the association matrix Q is (0, 1), $v \times v$ symmetric with zero trace, with constant row sums n_1 and has at most 3 distinct eigenvalues. Equivalently Q is the adjacency matrix of a strongly regular graph [4].

The study of certain two-class partially balanced designs—called special partially balanced incomplete block designs (SPBIBDs) arose in a natural manner in an earlier investigation on graphs and designs [3]. An SPBIBD of type $(\bar{s}, \bar{t})$ is a two-class PBIBD with the additional feature that there exist non-negative constants $\bar{s}$ and $\bar{t}$ such that for any treatment-block pair (x, B), the number of first associates of x in the block B is $\bar{s}$ or $\bar{t}$ depending on whether $x \in B$ or $x \notin B$. This is equivalent to the assertion

$$QN = (\bar{s} - \bar{t})N + \bar{t}J. \tag{7}$$

The following was established in [3].

Theorem 1.1. *Let D be a* PBIBD $(v, b, r, k, n_1, \lambda_1, \lambda_2)$ *based on the association matrix Q. Let $\theta = (\lambda_2 - r)/(\lambda_1 - \lambda_2)$. Suppose $\Omega(Q)$ denotes the spectrum of Q. If*

(i) $\theta \in \Omega(Q) - \{n_1\}$ *and Q is connected or*

(ii) $\theta = n_1$ *and* $|\Omega(Q)| = 2$,

then D is an SPBIBD.

We observe that SPBIBDs form a subclass of partial geometric designs. The purpose of this note is to investigate the converse situation. We shall find necessary and sufficient conditions on a partial geometric design $D(r, k, t, c)$ so that it is an SPBIBD of type $(\bar{s}, \bar{t})$. This is the content of our main result Theorem 2.2. As a consequence of this, we recover a result of [2] concerning partial geometric designs and balanced incomplete block designs (BIBDs).

2. Main result

The following Lemma relates SPBIBDs with partial geometric designs.

Lemma 2.1. *Let D be an SPBIBD of type $(\bar{s}, \bar{t})$ with parameters $(v, b, r, k, n_1, \lambda_1, \lambda_2)$. Then D is a partial geometric design (r, k, t, c) with*

$$t = g\bar{t} + \lambda_2 k, \qquad c = g\bar{s} + (\lambda_2 - 1)(k-1),$$

where $g = \lambda_1 - \lambda_2$.

Proof. Let N and Q denote respectively the incidence matrix and association matrix of D. Then we have

$$NJ = rJ, \qquad JN = kJ, \qquad NN^t = (r - \lambda_2)I + gQ + \lambda_2 J \tag{8}$$

and

$$QN = (\bar{s} - \bar{t})N + \bar{t}J. \tag{9}$$

Then,

$$NN^tN = [(r - \lambda_2) + g(\bar{s} - \bar{t})]N + (g\bar{t} + \lambda_2 k)J. \tag{10}$$

Comparing (2) and (10), we have the required result.

We now come to the converse situation.

Theorem 2.2. *Let $D(r, k, \bar{t}, c)$ be a partial geometric design with $v \times b$ incidence matrix N. Suppose the following conditions hold:*

(i) $t = g\bar{t} + \lambda_2 k$,

(ii) $c = g\bar{s} + (\lambda_2 - 1)(k-1)$,

(iii) $n_1 g = r(k-1) - \lambda_2(v-1)$ *and* $\bar{s}r = (g + \lambda_2)n_1$, *where $\bar{s}$, $\bar{t}$, λ_2, n_1 are non-negative integers and g an integer with $g + \lambda_2$ non-negative.*

(iv) *g divides $x_{ij} - \lambda_2 (i \neq j)$, where $X = (x_{ij}) = NN^t$. Then, D is an SPBIBD of type $(\bar{s}, \bar{t})$ with parameters $(v, b, r, k, n_1, \lambda_1 = g + \lambda_2, \lambda_2)$ based on the association scheme for which Q is $gQ = NN^t - (r - \lambda_2)I - \lambda_2 J$.*

Proof. Suppose conditions (i) through (iv) hold.

Put

$$X = NN^t \tag{11}$$

$$Y = X - (r - \lambda_2)I - \lambda_2 J. \tag{12}$$

Then, Y is a symmetric, integral matrix of valency $m_1 = gn_1$, with 0's along the main diagonal. Note that if $g = 0$, *then* $NN^t = (r - \lambda_2)I + \lambda_2 J$ and D is a BIBD (v, b, r, k, λ_2). In this case D is an SPBIBD of type $(k-1, k)$. Henceforth assume $g \neq 0$.

Put

$$Q = \frac{1}{g} Y. \tag{13}$$

Then (12), (13) and condition (iv) imply that Q is an integral matrix of valency n_1 with 0's along the main diagonal.

Next,

$$X^2 = NN^tNN^t = (NN^tN)N^t = [(r+k+c-1-t)N + tJ]N^t$$

gives

$$X^2 = (r+k+c-1-t)X + rtJ. \tag{14}$$

Then (i) and (ii) give

$$X^2 = [r-\lambda_2 + g(\bar{s}-\bar{t})]X + r(g\bar{t} + \lambda_2 k)J. \tag{15}$$

Now, using (12) and (15), we get

$$Y^2 = [g(\bar{s}-\bar{t}) + \lambda_2 - r]X + (r-\lambda_2)^2 I$$
$$+ [r(g\bar{t} + \lambda_2 k) + 2\lambda_2(r-\lambda_2) - 2\lambda_2 rk + \lambda_2^2 v]J. \tag{16}$$

Using (i)–(iii) and equating diagonal elements in (16) gives

$$(Y^2)_{ii} = g^2 n_1. \tag{17}$$

Next, $Q = (1/g)Y$, gives

$$QJ = \frac{1}{g}YJ = \frac{1}{g}n_1 gJ = n_1 J. \tag{18}$$

Then, (17) yields

$$(Q^2)_{ii} = n_1. \tag{19}$$

Thus, Q is an integral matrix satisfying trace $Q^2 = vn_1$ = sum of all the elements of Q. Hence Q is a (0, 1)-matrix. We thus observe that conditions (i) through (iv) imply that

$$NN^t = (r-\lambda_2)I + gQ + \lambda_2 J \tag{20}$$

and

$$QN = (\bar{s}-\bar{t})N + \bar{t}J \tag{21}$$

where Q is a symmetric, (0, 1)-matrix with zero trace and constant row sum n_1. From [1], we know that NN^t has at most 3 distinct eigenvalues. Then (20) implies that Q has at most 3 distinct eigenvalues. Thus Q is the adjacency matrix of a strongly regular graph and hence D is an SPBIBD of type $(\bar{s}, \bar{t})$. This completes the proof of Theorem 2.2.

Remark 2.3. Lemma 2.1 shows that conditions (i) and (ii) of Theorem 2.2 are satisfied by an SPBIBD of type $(\bar{s}, \bar{t})$. The first part of condition (iii) and condition (iv) are obviously satisfied. The second part of condition (iii) reduces to the parameter relation $\lambda_1 n_1 = \bar{s}r$, which is satisfied by SPBIBDs [3]. Thus, conditions (i) through (iv) are necessary and sufficient for a partial geometric design (r, k, t, c) to be an SPBIBD of type $(\bar{s}, \bar{t})$.

Corollary 2.4. [2, Theorem 4.1]. *Any partial geometric design* (r, k, t, c) *with*

$t=\lambda k$, $c=(k-1)(\lambda-1)$ *is a* BIBD (v, b, r, k, λ), *where*

$$v=\frac{r(k-1)}{\lambda}+1,\ b=\frac{vr}{k}.$$

Proof. Put $\lambda_1=\lambda_2=\lambda$, $g=0$ in Theorem 2.2.

Example 2.5. Let $X=\{1,2,\ldots,n\}$. Let P be the set of all unordered triplets from X. We call two distinct elements of P first, second or third associates if the corresponding triplets have two, one or zero symbols in common respectively. It can be easily checked that we obtain a three-class association scheme with parameters

$$n_1=3(n-3),\qquad n_2=3\binom{n-3}{2},$$

$$p_{11}^1=n-2,\qquad p_{11}^2=4,\qquad p_{11}^3=0,$$

$$p_{12}^1=2(n-4),\qquad p_{12}^2=2(n-4),\qquad p_{12}^3=9,$$

$$p_{13}^1=0,\qquad p_{13}^2=n-5,\qquad p_{13}^3=3(n-6).$$

We construct a configuration D by taking the elements of P as points and the set of all triplets containing a fixed symbol as the blocks. It can be verified that D is a partial geometric design with $v=\binom{n}{3}$, $b=n$, $r=3$, $k=\binom{n-1}{2}$, $t=3(n-2)$, $c=2(n-3)$. Moreover, D is a three-class PBIBD with parameters $(v, b, r, k, n_1, n_2, \lambda_1=2, \lambda_2=1, \lambda_3=0)$.

References

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